A STUDY ON DEFECTIVE COLOURING OF GRAPHS

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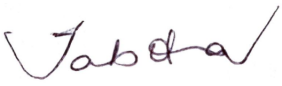
3. Dr Paradeshi Tabitha Rajashekar (RAC Member 3) Signature : 

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Abstract

If different colours represent different technology types that has to be placed on different geographical location, which can be denoted as the vertices of a graph, then the question on the possibility of placing the different technology type on the vertices such that there is minimum adjacency between the same technology type is a prominent study in research. This can be attained by maximising the properness of the colouring. However, if the number of technology type which represents the colours to colour the graph properly is not available, the properness of the colouring can be defined by certain colouring rules as per which, few of the colour classes are allowed to have adjacency between the elements it in. And the objective of defining such a colouring is to minimise the number of edges whose end vertices receive the same colour resulting from such colourings. An edge uv is said to be a bad edge if the colour assigned to both u and v are the same. A near proper colouring is a colouring that minimises the number of bad edges by permitting few colour classes to have adjacency between the elements in it. The $\delta^{(k)}$-colouring is a near proper colouring of G consisting of k given colours where $1 \leq k \leq \chi(G)-1$, which minimises the number of bad edges by permitting at most one colour class to have adjacency among the vertices in it.

The concerned study is the further work on the $\delta^{(k)}$-colouring of graphs. Few results on the $\delta^{(k)}$-colouring of any graph G have been investigated. The optimal $\delta^{(k)}$-colouring of various graph classes, derived graphs, graph products, graph powers that results in minimum number of bad edges obtained from the same for each different values of k where $1 \leq k \leq \chi(G) - 1$ is also determined.

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1 Introduction

Graph theory is a platform for the exploration of proof techniques in discrete mathematics, and its results have applications in many areas of computing, social, and natural sciences. The problem that is often said to have been the birth of graph theory is the famous *The Königsberg Bridge problem*. Graph theory has experienced explosive growth, due in large measure to its role as an essential structure underpinning modern applied mathematics. Computer science and combinatorial optimisation, in particular, draw upon and contribute to the development of the theory of graphs. Moreover, in a world where communication is of prime importance, the versatility of graphs makes them indispensable tools in the design and analysis of communication networks.

Intuitively, a graph consists of a set of points and a set of lines such that each line joins a pair of points if there is some relation between the two points. The set of points are commonly called as *nodes* or *vertices* and the lines are called as *edges* of a graph. The graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, disjoint from $V(G)$, together with an incidence function $I(G)$ that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $I(G(e)) = \{u, v\}$, then e is said to join u and v, and the vertices u and v are called the end vertices of the edge e. The number of vertices and edges in G are usually denoted by n or $v(G)$ or p and m or $e(G)$ or q respectively; these two basic parameters are called the order and size of G, respectively.

1.1 Need for the study

Graph theory, due to its vast applications and solutions for many real life problems, have become an area and ground for extensive research from past one hundred years. Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics

in computer science, physical, biological and social systems. Many problems of practical interest can be represented by graphs. In general graph theory has a wide range of applications in diverse fields. The travelling salesman's problem, the least spanning tree in a weighted graph, the optimal matching of jobs with persons, the channel assignment problems are few among the many examples in which graph theory is utilised in optimisation techniques.

Numerous problems of pure mathematics and theoretical computer science require the study of proper colourings and even more real-life problems require the calculation or at least an estimation of the chromatic number.

2 Review of the Literature

The Königsberg Bridge problem is a very well known problem whose solution ascribes to the beginning of graph theory. The solution for this problem was given by the Swiss Mathematician Leonhard Eüler in 1736. The Königsberg Bridge problem was an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island but without crossing any bridge twice. Euler argue that no such path exists. His proof involves only references to physical arrangement of the bridges, but eventually he proved the first theorem in Graph Theory. The first full-length book on the subject and the first comprehensive treatise on graph theory was considered to be the book written by Dénes Köing in 1936 [1].

2.1 Graph Colouring

Beginning with the origin of the Four Colour Problem in 1852, the field of graph colourings have developed into one of the most popular areas of graph theory. Though, its inception was with a question from a graduate student of arts and law, Francis Guthrie, who while colouring the map of England, observed that he could colour it with just four colours in such a way that no two countries sharing the

same line (boundaries) are coloured with a same colour, graph colouring has paved many solutions for numerous theoretical and practical problems and have made a strong foundation for great research in graph theory till the date. The Four colour conjecture was studied by many eminent researcher for over the following century (see, for instance [2]) before, finally, being answered in the affirmative by Appel and Haken [3, 4], in 1976. Chromatic graph theory has always contributed vital and fascinating research content to the ever curious minds of graph theorists. There are numerous variants of colourings and related problems. A tremendous number of literature is available for the different variants of graph colouring especially chromatic graph colouring (see, for instance [5, 6])

A graph colouring is an assignment of colours to the vertices or the edges of a graph, sometimes, subjected to certain conditions or rules. Now, a proper colouring of a graph is the assignment of colours to the vertices of a graph, unless not specifically mentioned, in such a way that the end vertices of an edge incident on them should be given different colours. The minimum number of colours required to colour a graph G properly is called the chromatic number of a graph, denoted by $\chi(G)$. A set $\{c_i \in C : c(v) = c_i \text{ for at least one } v \in V(G)\}$ is called a *colour class* of the colouring of G. A set U of vertices in a graph G is independent if no two vertices in U are adjacent. The maximum number of vertices in an independent set of vertices of G is called the vertex independence number, or more simply, the independence number of G and is denoted by $\alpha(G)$. By the definition of independence, it is clear that all the colour classes in a proper colouring must be an independent set.

The four colour problem was unsolved for more than a century and a tremendous research was done in solving the same. Due to which, many methods and tools were developed to find a solution for the problem that fascinated many mathematicians. In 1912, Birkhoff came up with a determinant formula for the number of ways of colouring a map. He proved that the number of ways of colouring the given map M in λ colours ($\lambda = 1, 2, \ldots$) is given by a polynomial $P(\lambda)$ of degree n, where n

is the number of regions of the map. This polynomial in λ defined for all real and complex values of λ was called chromatic polynomial of M. If it is proved that this polynomial has some roots for all maps M, then the famous problem is solved. Even though Birkhoff's assumption was not sufficient for an affirmative proof, a steady stream of attention was gained concerning the location of roots [7]. There is a lot of literature available on the chromatic polynomials of graphs (see [8]), to the extent that there is a book on this topic which summarises most of the research work on chromatic polynomials, see [9] and a chapter in a book, see [10].

2.2 Improper Colouring

Improper colouring also known as *defective colouring* or *relaxed colouring* was introduced almost simultaneously by Andrew and Jacobson (see [11]), Harary and Jones [12] and Cowen, Cowen and Woodall [13], and has continued to be an active area of research. Since the early nineties, a lot of work has been devoted to various aspects of improper colourings, both from a purely theoretical point of view [13–17] and in relation with frequency assignment issues [18–20]. A graph is $(k; d)$-colourable if one can colour the vertices with k colours such that no vertex is adjacent to more than d vertices of its same colour. In [13] Cowen, Cowen and Woodall focused on graphs embedded on surfaces and gave a complete characterisation of all k and d such that every outerplanar graph is $(k; d)$-colourable and a complete characterisation of all k and d such that every planar graph is $(k; d)$-colourable. Poh [21] and Goddard [22] showed that any planar graph has a special $(3, 2)$ -colouring in which each colour class is a linear forest (thus each colour class is the disjoint union of paths). The interested reader is also referred to the excellent book of Jensen and Toft [[6], cf. pages 40,63] for related problems and references. For general surfaces, it was shown in [13] that for each genus $g \geq 0$; there exists a $k = k(g)$ such that every graph on the surface of genus g is $(4, k)$-colourable. This was improved to $(3, k)$-colourable by Archdeacon [23]. For general graphs, a result of Lovász from

the 1960s [24], which has been rediscovered many times since (cf. [25–27]), provides an upper bound on the defective chromatic number of a graph. In [28], the result on the plane (see [13]) was extended to the torus. They showed that any graph embeddable on the torus is $(3, 2)$-colourable. Then it was shown that any such graph is also $(5, 1)$-colourable. Few of the notable works on acyclic improper colouring can be seen in [29–31]. A different generalisation of the chromatic number of a graph G was studied in 2004 by Charles D *et al.* called as k-clique-relaxed chromatic number, denoted by $\chi^{(k)}(G)$ (see [32]). Many engrossing recent work on relaxed colouring can be seen in [33–35]. In 2011, C Bujtas *et al.* came up with a different variation of improper colouring called as *improper C-colourings of graphs* [36]. For an integer $k \geq 1$, the k-improper upper chromatic number $\overline{\chi}_{k-imp}(G)$ of a graph G was introduced as the maximum number of colours permitted to colour the vertices of G such that, for any vertex v in G, at most k vertices in the neighbourhood $N(v)$ of v receive colours different from that of v.

An edge that joins two vertices which are coloured with a same colour is called *bad edge* [37]. In other words, an edge uv of a graph G is said to be bad if $c(u) = c(v)$. The k-defect polynomial $\phi_k(G; \lambda)$; counts the number of ways of colouring G with k bad edges. Thus, the chromatic polynomial of a graph G, $\chi(G; \lambda)$; is the 0-defect polynomial, $\phi_0(G; \lambda)$. E G Emphako Banda and J Kok in 2018, studied the properties of k-defect polynomial and highlighted the properties of chromatic polynomial which also applies for the k-defect polynomial (see [38]). The concept of near proper colouring was introduced in 2018 by J Kok and N K Sudev (see [39]). A *near proper colouring*, variation of defective colouring, is a colouring which minimises the number of bad edges by restricting the number of colour classes that can have adjacency among their own elements. From this concept of near proper colouring, they further discussed a particular case of near proper colouring which is called as $\delta^{(k)}$-colouring. The *$\delta^{(k)}$-colouring* is a near proper colouring of G consisting of k given colours where $1 \leq k \leq \chi(G) - 1$, which minimises the number of bad edges by permitting at most one colour class to have adjacency among the vertices in it. The

number of bad edges which results from a $\delta^{(k)}$-colouring of G is denoted by, $b_k(G)$. The $\delta^{(k)}$-colouring of path graph and in general bipartite graphs, cycle graph, and cycle related graphs like complete graph, wheel graph, and helm graph are studied and all possible ways of $\delta^{(k)}$-colouring is also discussed. Further, the minimum bad edges resulting from $\delta^{(k)}$-colouring of union of two graphs, join of two graphs and bounds for corona of two graphs is also determined (see [39]). It is clear that, the notion of $\delta^{(k)}$-colouring is a derivative of k-defect colouring. The critical difference is that for a $\delta^{(k)}$-colouring the number of colours, denoted by k, is prescribed and the condition of a minimum number of bad edges resulting from the k-colouring is set. In the k-defect polynomial, the number of colours denoted by λ and the number of bad edges, denoted by k, are prescribed. The notational interchange can be easily done.

3 Research Objectives and Methodology

The research work is the further study on the concept of $\delta^{(k)}$-colouring of graphs. As the concept of $\delta^{(k)}$-colouring of a graph was just introduced in 2018 and not much research work was done in the same, from a thorough study of the past literature, the following are the research objectives which elicit this research work.

1. To determine some general results on $\delta^{(k)}$-colouring of any graph G and determine the bounds for the minimum number of bad edges, $b_k(G)$, obtained from the $\delta^{(k)}$-colouring.

2. To investigate the $\delta^{(k)}$-colouring of few cycle related graph classes and find the optimal $\delta^{(k)}$-colouring that gives minimum number of bad edges for different values of k.

3. To work on the minimum number of bad edges obtained from $\delta^{(k)}$-colouring of generalised Petersen graph using the concept of independence number of graph, $\alpha(G)$.

4. To bring forth the optimal $\delta^{(k)}$-colouring of few powers of graphs viz. path graph, cycle graph, helm graph and closed helm graph by running through all the possible $\delta^{(k)}$-colourings of the same and investigating the minimum number of bad edges obtained from $\delta^{(k)}$-colouring for each different values of k.

5. To find out the minimum number of bad edges for each different possible values of k resulting from $\delta^{(k)}$-colouring by determining a quintessential $\delta^{(k)}$-colouring of some derived graphs of graphs like Mycielski graph of graph, splitting graph of graphs, shadow graph of graphs and couple of products of graphs viz. Cartesian product, direct product and corona product of two graphs.

   The research mainly aims in focusing on the minimum number of bad edges resulting from $\delta^{(k)}$-colouring for not just one or two values of k but for all the different possible values of k where $1 \leq k \leq \chi(G) - 1$. This was attained by finding an optimised $\delta^{(k)}$-colouring by exhausting all possible ways of $\delta^{(k)}$-colouring of the particular graph classes and referring to the proof techniques from the literature such as direct proof method, method of contradiction etc. The research objectives mentioned above were accomplished by following a well organised plan where in every four to six months an objective was settled.

The content of the thesis entitled “A study on defective colouring of graphs" has been divided into six chapters which are briefly given below.

4 Introduction

This chapter is an overview of the history of graph theory and the basic terminologies, definitions, concepts etc. that are necessary for the investigation concerned. The literature review of the past works related to the study concerned is also included in this chapter.

For all terms and definitions, refer to [40–42] and for graph classes, see [43, 44]. Further, for graph colouring, see [45–47]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

5 On the $\delta^{(k)}$-Colouring of Graphs

In this chapter, few of the general results for the $\delta^{(k)}$-colouring of any graph G are investigated and the bounds for the minimum number of bad edges, $b_k(G)$, obtained from $\delta^{(k)}$-colouring of G are also determined. This chapter also focuses on obtaining the minimum number of bad edges resulting from $\delta^{(k)}$-colouring of generalised Petersen graphs using the concept of independence number of graph, $\alpha(G)$. Further, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring of some derived graphs viz. Mycielski graph of a graph $\mu(G)$, splitting graph of a graph $S(G)$ and shadow graph of a graph $SH(G)$ are studied. The graph classes considered for the $\delta^{(k)}$-colouring of splitting graph and shadow graph of a graph are cycle graph C_n, wheel graph $W_{1,n}$ and complete graph K_n. The result obtained for the $\delta^{(k)}$-colouring of Mycielski graph of a graph is determined for any graph G.

Definition 5.1. A *near proper colouring* is a colouring which minimises the number of bad edges by restricting the number of colour classes that can have adjacency among their own elements [39].

Definition 5.2. [39] For $1 \leq k < \chi(G)$, the $\delta^{(k)}$*-colouring* is a near proper colouring of G consisting of k given colours, which minimises the number of bad edges by permitting at most one colour class to have adjacency among the vertices in it. The number of bad edges obtained from $\delta^{(k)}$-colouring is denoted by $b_k(G)$.

In view of the above notion, the $\delta^{(k)}$-colouring of path graph, bipartite graphs, cycle graph and few cycle related graphs like wheel graph, helm graph, complete graphs and the union, join and the bound for the corona of any two graphs G and

H are determined in [39]. The following are the results on the $\delta^{(k)}$-colouring of any graph G.

Theorem 5.3. *Let G be any graph and H be its induced subgraph then $b_k(G) \geq b_k(H)$.*

Theorem 5.4. *Let G be any graph and $H_1, H_2, \ldots, H_r$ be the vertex disjoint induced subgraphs of the graph G then $b_k(G) \geq b_k(H_1) + b_k(H_2) + \ldots + b_k(H_r)$.*

Theorem 5.5. *Let G be any graph with n vertices and m edges, then the number of bad edges in G is given as,*

$$1 \leq b_k(G) \leq \frac{(n-k+1)(n-k)}{2}.$$

Theorem 5.6. *Let G be any graph of size m, then the number of bad edges in G for $k \geq 2$ is given as,*

$$b_k(G) \leq m - 2.$$

Theorem 5.7. *For any graph G, the number of bad edges resulting from a $\delta^{(k)}$-colouring of the graph with k_1 and k_2 colours is given as $b_{k_1}(G) > b_{k_2}(G)$ when $k_1 < k_2$ where $1 \leq k_1 < k_2 \leq \chi(G) - 1$.*

Theorem 5.8. *For any graph G, if $b_k(G) = 1$ then $k = \chi(G) - 1$.*

Remark 1. Note that, the converse for the above theorem is not true. There are so many counter examples for the Theorem 5.8, one such example is given as below.

For a Petersen graph, the chromatic number $\chi(P(5,2))$ is 3. Thus, in a $\delta^{(k)}$-colouring, when the Petersen graph $P(5,2)$ is coloured with $k = \chi(G) - 1 = 2$ colours, it leads to 3 bad edges.

Further, this chapter discusses the $\delta^{(k)}$-colouring of generalised Petersen graph $P(n,t)$ and determines the number of bad edges obtained from the same with the help of its independence number $\alpha(P(n,t))$. Note that, the chromatic number of a

generalised Petersen graph $P(n,t)$ is 3 and hence the concern study considers the $\delta^{(k)}$-colouring of $P(n,t)$ with two colours, say c_1 and c_2 only. It can also be noted that, the $\delta^{(2)}$-colouring of the graph $P(n,t)$ depends on its independence number. For the independence number of $P(n,t)$, refer to [48,49]. It is also to be noted that when n is even and t is odd, the graph $P(n,t)$ is bipartite and hence will have no bad edges when $k=2$. Hence, in the following discussion, the other possibilities only are considered.

Since, $k=2$, the minimum number of bad edges is obtained when the use of colour c_2 is maximised and by the assumption, the colour class C_2 should be an independent set. Thus, the $\alpha(P(n,t))$ number of vertices are assigned the colour c_2 and remaining vertices the colour c_1. Now, when $t=1$, it is known that, the $\alpha(P(n,1))$ is n or $n-1$ when n is even or odd respectively (see [48,49]). As mentioned above $P(n,t)$ is bipartite when n is even and t is odd, hence the case when n is odd and $t=1$ are only considered. Since, $\alpha(P(n,1))=n-1$ when n is odd, the $n-1$ vertices of $P(n,1)$ can be coloured with c_2 and the remaining with c_1. Hence, the total number of bad edges in $P(n,1)$ is $|E(P(n,1))|-3(n-1)=3$.

Proposition 5.9. *For $P(n,2)$ where $n\geq 5$, $b_2(P(n,2))=3\lceil\frac{n}{5}\rceil$.*

Proposition 5.10. *For $P(n,3)$ where n is odd and $n>6$, $b_2(P(n,3))=6$.*

Proposition 5.11. *For $P(n,5)$ where n is odd and $n>10$, $b_2(P(n,5))=9$.*

From the above two cases, a fixed pattern followed for $P(n,t)$ where both n and t are odd can be observed. But, it is not true that the number of bad edges in the graph $P(n,t)$; n and t are odd, follows this fixed pattern. Hence, finding the bounds for the number of bad edges in $P(n,t)$; n and t are odd, is interesting and significant. The other subclasses of the generalised Petersen graphs are also considered in the discussion which are as followings.

Theorem 5.12. *For $P(n,t)$ where n and t are odd and $n>6$, $b_2(P(n,t))\geq\frac{3(t+1)}{2}$.*

Remark 2. Note that, the independence number for $P(n,t) = \frac{2n-t-1}{2}$ if n and t are odd and t divides n. Hence, the equality for the above theorem holds when t divides n.

Theorem 5.13. *If n, t are both even integers, then,*

$$b_2(P(n,t)) \geq \frac{3}{2}\left(n + d\left\lfloor \frac{n}{2d} \right\rfloor\right) \approx \frac{3n}{4},$$

where $d = gcd(n,t)$.

Theorem 5.14. *For any integer $t \geq 1$, the total number of bad edges is given by*

$$b_2(P(3t,t)) = 9t - 3\left\lceil \frac{5t-2}{2} \right\rceil.$$

Theorem 5.15. *If n and t are integers where n is odd and t is even, the number of bad edges is given by*

$$\begin{aligned} b_2(P(n,t)) &\geq 3\left(n - \left(\frac{n-1}{2}\right) - \left\lfloor \frac{\lceil \frac{n}{t} \rceil + 1}{2} \right\rfloor \left\lfloor \frac{n}{2d\lceil \frac{n}{t} \rceil} \right\rfloor - \left(\frac{d-1}{2}\right)\left\lfloor \frac{1}{2}\left(\frac{n}{d} \pmod{\lceil \frac{n}{t} \rceil}\right) \right\rfloor\right) \\ &\approx \frac{3(2n - (\frac{n}{d}) - 2d + 2)}{4} \end{aligned}$$

where $d = \gcd(n,t)$.

Theorem 5.16. *If m and l are positive integers with m odd, then*

$$b_2(P(2ml \pm 1, 2l)) \geq 3\left((2ml \pm 1) - \left(\frac{2(2ml \pm 1) - m - 1}{2}\right)\right)$$

Furthermore,

$$b_2(P(2^p l \pm 1, 2^{p-1} l)) = 3\left((2^p l \pm 1) - \left\lfloor \frac{4(2^p l \pm 1)}{5} \right\rfloor\right).$$

Now from all the theorems and discussions mentioned above for $P(n,t)$, a general result for the bad edges obtained from $\delta^{(k)}$-colouring of any r-regular graph G, for

the minimum value of k which is 2, when the independence number $\alpha(G)$ is known, is given as follows.

Theorem 5.17. *Let G be a r-regular graph and $\alpha(G)$ be its independence number, then we have $b_2(G) = \frac{r(n-2\alpha)}{2}$.*

A derived graph of a graph G is a graph that is obtained from the graph G by adding few vertices and few conditions to the adjacency of the vertices in the derived graph. The properties of various concepts of derived graph of a graph are well studied in the literature. In the below discussion, the $\delta^{(k)}$-colouring and the number of bad edges obtained from it for Mycielski graph of graphs, splitting graph of graphs and shadow graph of graphs for the basic graphs namely cycle graphs, wheel graphs and complete graphs are studied. The $\delta^{(k)}$-colouring of the above mentioned derived graph of path graphs are not studied as the graph is a bipartite graph. Now, since for all the above mentioned derived graphs of the graphs, the underlined graph G is a subgraph, there can be relations between the number of bad edges in G and the number of bad edges in the derived graph of G obtained from $\delta^{(k)}$-colouring of G as well as $\delta^{(k)}$-colouring of derived graph of G. Or, in other words it can be said that $b_k(G)$ will be less than or equal to the number of bad edges resulting from a $\delta^{(k)}$-colouring of its derived graph.

Definition 5.18. From a simple graph G, Mycielski construction produces a simple graph $G^{'}$ containing G. Beginning with G having vertex set $V = \{v_1, v_2, \ldots, v_n\}$ add vertices $U = \{u_1, u_2, \ldots, u_n\}$ and one more vertex w. Add edges to make u_i adjacent to all of $N_G(v_i)$, and finally let $N(w) = U$ [42].

Theorem 5.19. *For the Mycielski graph of graphs the number of bad edges is given by, $b_k(\mu(G)) = b_k(G) + \theta(c_1) + \eta$, where η is the number of edges between U and V that are incident on the end vertices of the bad edges in G and $\theta(c_1)$ is the number of times a colour c_1 is allocated to the vertices of $\mu(G)$.*

Theorem 5.20. *For a Mycielski graph $\mu(G)$ of a graph G, if the number of bad edges obtained from a $\delta^{(k)}$-colouring is 1 then $k = \chi(\mu(G)) - 1$.*

Definition 5.21. For each vertex v of a graph G, take a new vertex v'. Join the vertex v' to all the vertices of G adjacent to the vertex v. The graph $S(G)$ thus obtained is called *splitting graph* of G (see [50]).

In the following results, the $\delta^{(k)}$-colouring of splitting graph of cycle graph $S(C_n)$, wheel graph $S(W_{1,n})$ and complete graph $S(K_n)$ are discussed and the number of bad edges obtained from the $\delta^{(k)}$-colouring for the same have been studied.

Theorem 5.22. *For a splitting graph $S(G)$ of a graph G, if the number of bad edges obtained from a $\delta^{(k)}$-colouring is 3 when $b_k(G)$ is 1 then $k = \chi(S(G)) - 1$.*

Theorem 5.23. *For a splitting graph of a cycle $S(C_n)$ where n is odd, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by, $b_2(S(C_n)) = 3$.*

Theorem 5.24. *For a splitting graph of a wheel $S(W_{1,n})$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(S(W_{1,n})) = \begin{cases} \frac{3n}{2}, & \text{if } n > 3 \text{ is even and } k = 2, \\ 2n, & \text{if } n < 9 \text{ is odd and } k = 2, \\ \frac{3(n+3)}{2}, & \text{if } n \geq 9 \text{ is odd and } k = 2, \\ 3, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Theorem 5.25. *For a splitting graph of a complete graph $S(K_n)$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by, $b_k(S(K_n)) = \frac{3(n-k+1)(n-k)}{2}$.*

Definition 5.26. *The shadow graph $SH(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G'' [51].*

Now, since none of the u''s are adjacent to its corresponding u'''s, the colour assigned to each u' in G' can be given to its corresponding u'' in G''. Thus, the chromatic number of $SH(G)$ is equal to the chromatic number of G.

Theorem 5.27. *For a shadow graph of a cycle graph $SH(C_n)$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by, $b_2(SH(C_n)) = 4$.*

Theorem 5.28. *For a shadow graph of a complete subgraph $SH(K_n)$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by, $b_k(K_n) = 2(n-k+1)(n-k)$.*

Theorem 5.29. *For a shadow graph of a wheel graph $SH(W_{1,n})$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(SH(W_{1,n})) = \begin{cases} 2n, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ 2(n+3), & \text{if } n \text{ is odd and } k = 2, \\ 4, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Theorem 5.30. *For a shadow graph $SH(G)$ of graph G, if the number of bad edges obtained from a $\delta^{(k)}$-colouring is 4 when $b_k(G)$ is 1 then, $k = \chi(SH(G)) - 1$.*

6 On the $\delta^{(k)}$-Colouring of Certain Cycle Related Graphs

This chapter focuses on the $\delta^{(k)}$-colouring of certain cycle related graph like closed helm graph ($CH_{1,n}$), web graph ($Wb_{1,n}$), sun graph (S_n), closed sun graph (CS_n), antiprism graph (A_n), sunlet graph (Sl_n), generalised friendship graph ($F_{m,n}$), double wheel graph ($DW_{1,n}$), flower graph ($F_{1,n}$), sunflower graph ($SF_{1,n}$), closed sunflower graph ($CSF_{1,n}$), blossom graph ($Bl_{1,n}$), Djembe graph ($Dj_{1,n}$). Among the basic graph classes, the path graph and cycle graphs are the most fundamental graph classes. The $\delta^{(k)}$-colouring of cycle graph and a couple of cycle related graphs are already discussed in [39]. Now, the following discussion is about few of the cycle related graph. The $\delta^{(k)}$-colouring of each graph class is obtained for different values of k by going through all the possible $\delta^{(k)}$-colourings for each graph classes and the

optimum $\delta^{(k)}$-colouring that gives the minimum number of bad edges resulting from the same is determined. Also, it is clear that when $k = 1$, the minimum number of bad edges obtained from $\delta^{(k)}$-colouring is $|E(G)|$ and hence, throughout the discussion, the value of k is considered to be $2 \leq k \leq \chi(G) - 1$. The following are the results obtained for different graph classes.

Definition 6.1. *[52] A closed helm graph, denoted by $CH_{1,n}$, is the graph obtained from a helm graph $H_{1,n}$ by joining its pendant vertices to form a cycle called an outer cycle.*

Theorem 6.2. *For a closed helm graph $CH_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(CH_{1,n}) = \begin{cases} \frac{n}{2}, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ \lceil \frac{n}{2} \rceil + 3, & \text{if } n \geq 5 \text{ is odd and } k = 2, \\ 1, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Definition 6.3. *[53] A web graph, denoted by $Wb_{1,n}$, is the graph obtained by attaching one pendant edge to each vertex of the outer cycle of the closed helm graph $CH_{1,n}$.*

Theorem 6.4. *For a web graph $Wb_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(Wb_{1,n}) = \begin{cases} \frac{n}{2}, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ \lceil \frac{n}{2} \rceil + 3, & \text{if } n \geq 5 \text{ is odd and } k = 2, \\ 1, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Definition 6.5. *[52] A double-wheel graph $DW_{1,n}$ is the graph obtained by connecting the vertices of two (disjoint) cycles each of size n to a common vertex called central vertex. That is, $DW_{1,n} := 2C_n + K_1$.*

Theorem 6.6. *For a double wheel graph $DW_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(DW_{1,n}) = \begin{cases} n, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ 2(\lceil \frac{n}{2} \rceil + 1), & \text{if } n \geq 3 \text{ is odd and } k = 2, \\ 1, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Definition 6.7. *[52] A flower graph, denoted by $F_{1,n}$, is the graph obtained from a helm graph $H_{1,n}$ by joining each of its pendent vertices to its central vertex.*

Theorem 6.8. *For a flower graph $F_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(F_{1,n}) = \begin{cases} n, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ n+1, & \text{if } n \geq 3 \text{ is odd and } k = 2, \\ 1, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Definition 6.9. *[52] A sunflower graph $SF_{1,n}$ is obtained from a wheel graph $W_{1,n}$ with the vertices $v_1, v_2, \ldots, v_n$ as the rim vertices and v_0 as its central vertex, by attaching n u_i's corresponding to each v_i where $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1}.*

Theorem 6.10. *For a sunflower graph $SF_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(SF_{1,n}) = \begin{cases} n, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ n, & \text{if } n \geq 3 \text{ is odd and } k = 2, \\ 1, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Definition 6.11. *[52] A closed sunflower graph $CSF_{1,n}$, is obtained by adding edge between each u_i and u_{i+1} of the sunflower graph.*

Theorem 6.12. *For a closed sunflower graph $CSF_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(CSF_{1,n}) = \begin{cases} \frac{3n}{2}, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ \frac{n}{2}, & \text{if } n \geq 4 \text{ is even and } k = 3, \\ \frac{5n-3}{2}, & \text{if } n \leq 7 \text{ is odd and } k = 2, \\ 2(n+1), & \text{if } n \geq 7 \text{ is odd and } k = 2, \\ \frac{n+5}{2}, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Definition 6.13. *[52] A Blossom graph $Bl_{1,n}$ is obtained by making each u_i adjacent to the central vertex of the closed sunflower graph.*

Theorem 6.14. *For a blossom graph $Bl_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_4(Bl_{1,n}) = \begin{cases} 2, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

$$b_3(Bl_{1,n}) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \pmod 3, \\ 2\lceil \frac{n}{3} \rceil + 3, & \text{if } n \equiv 1 \pmod 3, \\ 2\lceil \frac{n}{3} \rceil + 1, & f\, n \equiv 2 \pmod 3. \end{cases}$$

$$b_2(Bl_{1,n}) = \begin{cases} \frac{7n}{2}, & \text{if } n \text{ is even}, \\ \frac{7n-5}{2}, & f\, n \text{ is odd}. \end{cases}$$

Definition 6.15. *[54] A djembe graph $Dj_{1,n}$ is the graph obtained by joining the corresponding vertices two cycles of the same order n and joining all the vertices of the two cycles to an external vertex. That is, $Dj_{1,n} = (C_n \square P_2) + K_1$. This external vertex may be called the central vertex of the djembe graph.*

Theorem 6.16. *For a Djembe graph $Dj_{1,n}$, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(Dj_{1,n}) = \begin{cases} n, & \text{if } n \geq 4 \text{ is even and } k = 2, \\ n+4, & \text{if } n \geq 3 \text{ is odd and } k = 2, \\ 2, & \text{if } n \geq 3 \text{ is odd and } k = 3. \end{cases}$$

Definition 6.17. *The sun graph, S_n, is the graph obtained from adding n vertices $\{u_1, u_2, \ldots, u_n\}$ corresponding to each vertex of a complete graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$ such that each u_i is adjacent to its corresponding v_i and v_{i+1} [55].*

Theorem 6.18. *For a sungraph S_n, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(S_n) = \begin{cases} \frac{(n-k+1)(n-k)}{2}, & \text{if } k \geq 3 \text{ and } \forall\ n, \\ \\ \frac{n(n-3)+6}{2}, & \text{if } k = 2 \text{ and } \forall\ n. \end{cases}$$

Definition 6.19. *The closed sun graph, $\overline{S_n}$, is the graph obtained from adding the edges u_iu_{i+1} in the sun graph ([55]).*

Theorem 6.20. *For a closed sungraph S_n, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_k(CS_n) = \begin{cases} \frac{(n-k+1)(n-k)}{2}, & \text{if } k \geq 4 \text{ and } \forall\ n, \\ \\ \frac{(n-k+1)(n-k)}{2}, & \text{if } k = 3 \text{ and } n-3 \text{ is even}, \\ \\ \frac{n(n-3)+6}{2}, & \text{if } k = 3 \text{ and } n-3 \text{ is odd}, \\ \\ \frac{n(n-1)+6}{2}, & \text{if } k = 2 \text{ and } \forall\ n. \end{cases}$$

Definition 6.21. *Consider two cycles C_n and C_n of order n with vertex sets and $\{u_1, u_2, \ldots, u_n\}$, and $\{v_1, v_2, \ldots, v_n\}$ respectively. The antiprism graph; A_n is the graph obtained by joining the vertices of these cycles such that v_iu_i, $v_iu_{i+1} \in E(A_n)$ in the sense that $u_n + r = u_r$ ([55]).*

Theorem 6.22. *For an antiprism graph A_n, the number of bad edges obtained from a $\delta^{(k)}$-colouring is given by,*

$$b_3(A_n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3, \\ 2, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

$$b_2(A_n) = \begin{cases} 2n, & \text{if } n \text{ is even}, \\ 2(n-1), & \text{f } n \text{ is odd}. \end{cases}$$

Definition 6.23. *The sunlet graph Sl_n with $2n$ vertices and $2n$ edges, is obtained by attaching a pendant edge to each vertex of cycle of length n.*

For a sunlet graph Sl_n, the chromatic number is 2 when n is even and is 3 when n is odd. Since, Sl_n is a bipartite graph when n is even, it will have no bad edges when coloured with $k = 2$ colours and as mentioned above, throughout the study, the $\delta^{(1)}$-colouring of graphs are not considered and hence the case when $k = 2$ is considered when n is odd.

Theorem 6.24. *For a sunlet graph Sl_n when n is odd, the minimum number of bad edges obtained from the $\delta^{(2)}$-colouring is given by, $b_2(Sl_n) = 1$.*

The friendship graph F_n is a collection of n K_3's with a common vertex and a generalised friendship graph $F_{m,n}$ is a collection of n cycles of length m with a common vertex. Now, the chromatic number $\chi(F_{m,n})$ is 3 when m is odd and the graph is bipartite when m is even and hence, the $\delta^{(k)}$-colouring of $F_{m,n}$ when m is odd is discussed. Now, since $\chi(F_{m,n}) = 3$ when m is odd, the only value that k can take in this case is 2. Following given is the result for the minimum number of bad edges obtained from $\delta^{(k)}$-colouring of $F_{m,n}$ when m is odd.

Theorem 6.25. *For a generalised friendship graph $F_{m,n}$ when m is odd and $n \geq 1$, the minimum number of bad edges obtained from $\delta^{(k)}$-colouring is given as, $b_2(F_{m,n}) = n$.*

7 On the $\delta^{(k)}$-Colouring of Powers of Paths and Cycles

Among different classes of connected graphs, paths and cycles are the most elementary graph classes. In this chapter, the $\delta^{(k)}$-colouring and the optimum number of bad edges corresponding to this colouring of the powers of paths and cycles for all the possible values of k where $1 \leq k \leq \chi(G) - 1$ are discussed. Because of their structural characteristics, the powers of paths and cycles possess much importance in the study on $\delta^{(k)}$-colouring and the corresponding minimum bad edges. First, recall that the diameter of a path and cycle graph of order n is $n-1$ and $\lfloor \frac{n}{2} \rfloor$ respectively. The cases when $r = 1$ and, $r = n - 1$ and $r = \lfloor \frac{n}{2} \rfloor$ for both paths and cycles respectively have already been discussed in [39]. Hence, the main focus of the study concerned will be on the $\delta^{(k)}$-colouring of the r-th power of a path and the r-th power of a cycle C_n for $1 < r < n - 1$ and $1 < r < \lfloor \frac{n}{2} \rfloor$ respectively.

Theorem 7.1. *For $n \equiv m \pmod{r+1}$, the number of bad edges obtained from $\delta^{(k)}$-colouring in P_n^r where $2 \leq r \leq n-1$ and $k \leq r$ is,*

$$b_k(P_n^r) = \begin{cases} \frac{(r-k+2)(r-k+1)}{2}(2(\lfloor \frac{n}{r+1} \rfloor) - 1), & \text{if } 0 \leq m < k, \\ \frac{(r-k+2)(r-k+1)}{2}(2(\lfloor \frac{n}{r+1} \rfloor) - 1) + & \\ \frac{(m-k+1)(m-k))}{2} + \frac{m-k+1}{2}(2r - m - k + 2), & \text{if } m \geq k. \end{cases}$$

Theorem 7.2. *For C_n^r, where $2 \leq r < \lfloor \frac{n}{2} \rfloor$ and $2 \leq k < \chi(G)$ and $k \leq r$, the number of bad edges obtained from $\delta^{(k)}$-colouring is given by,*

$$b_k(C_n^r) = \begin{cases} (\frac{n}{r+1})(r-k+2)(r-k+1), & \text{if } n \equiv 0 \pmod{r+1}, \\ \frac{(r-k+2)}{2}\left((r-k+1)(2\lfloor \frac{n}{r+1} \rfloor - 1) + (r-k+3)\right), & \text{if } n \equiv 1 \pmod{r+1}. \end{cases}$$

Theorem 7.3. *For the powers of cycle C_n^r, where $2 \leq r < \lfloor \frac{n}{2} \rfloor$, the number of bad edges obtained from $\delta^{(k)}$-colouring when $n \equiv m \pmod{r+1}$ where $m \geq 2$ is given by,*

$$b_k(C_n^r) = ab + a(b-1) + c + d + e,$$

where,

$$a = \frac{(r-k+2)(r-k+1)}{2}$$

$$b = \left\lfloor \frac{n}{r+1} \right\rfloor$$

$$c = \begin{cases} r - m + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r-m < r-k+1 \text{ when } k > m, \\ r - k + 1 + \frac{(r-m-k)(r-m-k+1)}{2}, & \text{if } r-m \geq r-k+1 \text{ when } k \leq m. \end{cases}$$

$$d = \begin{cases} \frac{(r-k)(r-k+1)+2m}{2}, & \text{if } m \geq r-k, \\ \frac{m}{2}(2(r-k)-m+3), & \text{if } m < r-k, \\ m - r + k, & \text{if } m \leq k \text{ and if } m = k = r. \end{cases}$$

$$e = \begin{cases} m(r-k+1), & \text{if } m \leq k, \\ k(r-k+1) + \frac{m-k}{2}(2r-m-k+1), & \text{if } m > k. \end{cases}$$

Theorem 7.4. *For C_n^r, where $2 \leq r < \lfloor \frac{n}{2} \rfloor$ and $3 \leq k < \chi(G)$ and $k > r$, the number of bad edges obtained from $\delta^{(k)}$-colouring is given by,*

$$b_k(C_n^r) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{r+1}, \\ 1, & \text{if } n \equiv 1 \pmod{r+1}. \end{cases}$$

8 On the $\delta^{(k)}$-Colouring of Powers of Helm and Closed Helm Graph

A *wheel graph* $W_{1,n}$ is a graph obtained by drawing edges from all its vertices of a cycle to an external vertex. The square of a wheel graph is a complete graph and the number of bad edges of complete graphs obtained from a $\delta^{(k)}$-colouring have been already discussed in [39]. The two popular graph classes generated from wheel graphs are helm graphs and closed helm graphs. This chapter aims in finding the optimal $\delta^{(k)}$-colouring can results in minimum number of bad edges of the powers of these two wheel related graph classes.

Definition 8.1. *A helm graph $H_{1,n}$ is a graph obtained by attaching a pendant vertex to every rim vertex (vertex of the outer cycle) of a wheel graph $W_{1,n}$.*

Note that the diameter of a helm graph is 4 and hence it is necessary to determine the number of bad edges obtained from a $\delta^{(k)}$-colouring in its square and cube.

Theorem 8.2. *For $H^r_{1,n}$, where $r = 2$ and 3 and $2 \leq k < \chi(H^r_{1,n})$, the number of bad edges obtained from $\delta^{(k)}$-colouring is given by,*

$$b_k(H^r_{1,n}) = \begin{cases} \frac{(n-k+2)(n-k+1)}{2}, & \text{if } r = 2 \text{ and } n \geq 6, \\ \min(\frac{n^2+n}{2}, \frac{n^2-n+18}{2}), & \text{if } r = 2 \text{ and } k = 2, \\ \min(n(n+1), \frac{n(3n-1)}{2}, & \text{if } r = 3,\ k = 2 \text{ and } n \text{ is even}, \\ \min(\frac{n^2+n+2}{2} + \lceil \frac{n}{2} \rceil (n+1), \frac{n(3n-1)}{2}), & \text{if } r = 3,\ k = 2 \text{ and } n \text{ is odd}, \\ \frac{n(n-2k+7)}{2}, & \text{if } 3 \leq k \leq \chi(H_{1,n}) \text{ and } n \text{ is even}, \\ \frac{n(n-2k+9)}{2}, & \text{if } 4 \leq k \leq \chi(H_{1,n}) \text{ and } n \text{ is odd}. \end{cases}$$

A *closed helm graph* is graph obtained by joining the pendant vertices of a helm graph so as to form another (outer) cycle. For a closed helm graph, the diameter is 3 and 4 when $n = 6, 7$ and $n \geq 8$ respectively. Hence, the square of $CH_{1,n}$ for all $n \geq 6$ and the cube of $CH_{1,n}$ for all $n \geq 8$ are taken into consideration. Throughout the discussions, the u_i's where $1 \leq i \leq n$ are the vertices of the outer cycle (cycle joined by the pendant vertices of helm graph) and v_i's where $0 \leq i \leq n$ are the vertices of the inner wheel graph which forms a complete subgraph when taken the power. The colours $c_1, c_2 \ldots, c_k$ are the k available colours with their respective colour classes $C_1, C_2, \ldots, C_k$.

Having taken the square and cube of a closed helm graph, the outer cycle formed is the r-th power of C_n where $r = 2$ and $r = 3$ and the inner wheel graph is a complete graph of order $n + 1$. The powers of cycle graph is a $2r$ regular graph and every vertex is in complete subgraph of order $r + 1$. Now, any vertex can be assigned any colour only once and so the vertex disjoint complete subgraphs of C^r_n each of order $r + 1$ are considered. There are $\lfloor \frac{n}{r+1} \rfloor$ vertex disjoint complete subgraphs of order $r + 1$. Here, the vertex disjoint complete subgraphs in clock wise directions

(note that, the same in any direction is considered since it is a regular graph) are considered.

Theorem 8.3. *For $CH^2_{1,n}$ where $n \geq 6$, the number of bad edges obtained from $\delta^{(k)}$-colouring when $k = 2$ is given by,*

$$b_2(CH^2_{1,n}) = \begin{cases} \frac{n^2+9n+15}{2}, & \text{if } n \equiv 0 \pmod 3, \\ \frac{n^2-n+20\lfloor \frac{n}{3} \rfloor+6}{2}, & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Theorem 8.4. *For $2 \leq k < \chi(CH^3_{1,n})$ and $k \leq 3$, the number of bad edges obtained from $\delta^{(k)}$-colouring in $CH^3_{1,n}$ where $n \geq 8$ is given by,*

$$b_k(CH^3_{1,n}) = \begin{cases} \frac{n(7n-6k+12)+k(k-5)}{2}, & \text{if } n \equiv 0 \pmod 4, \\ \lfloor \frac{n}{4} \rfloor(5-k)(5-k+n) + n^2 + 3n + k + 6, & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Theorem 8.5. *For the square of closed helm graph $CH^2_{1,n}$ where $n \equiv 2 \pmod 3$ and $k = 2$, the number of bad edges is given by,*

$$b_k(CH^2_{1,n}) = \frac{n^2 - n + 10}{2} + 8\lfloor \frac{n}{3} \rfloor + b_k(C^2_n).$$

Theorem 8.6. *For the cube of closed helm graph $CH^3_{1,n}$, where $n \equiv m \pmod 4$ where $2 \leq m \leq 3$ and $k \leq 3$, the number of bad edges is given by,*

$$b_k(CH^3_{1,n}) = (n+1)(\frac{n}{2} + m + (r-k+2)\lfloor \frac{n}{4} \rfloor) + b_k(C^3_n).$$

Theorem 8.7. *For $3 \leq k < \chi(CH^2_{1,n})$, the number of bad edges obtained from $\delta^{(k)}$-colouring of $CH^2_{1,n}$ where $n \geq 6$ is given by,*

$$b_k(CH^2_{1,n}) = \begin{cases} \frac{n^2-\frac{n}{3}+16}{2}, & \text{if } k = 3 \text{ and } n \equiv 0 \pmod 3, \\ \frac{(n-k)^2+3(n-k+7)}{2}, & \text{if } k > 3 \text{ and } n \equiv 0 \pmod 3, \\ \frac{3n^2-n+12}{6}, & \text{if } k = 3 \text{ and } n \equiv 1 \pmod 3, \\ \frac{n^2-5n+8}{2}, & \text{if } k = 4 \text{ and } n \equiv 1 \pmod 3, \\ \frac{(n-k+2)(n-k+1)}{2}, & \text{if } k > 4,\ n \equiv 1 \pmod 3. \end{cases}$$

Theorem 8.8. *For $4 \leq k < \chi(CH^3_{1,n})$, the number of bad edges obtained from $\delta^{(k)}$-colouring in $CH^3_{1,n}$ is given by,*

$$b_k(CH^r_{1,n}) = \begin{cases} \frac{(n-k+5)(3n-2k+8)}{4}, & \text{if } n \equiv 0 \pmod 4, \\ (n+1)(\frac{n}{2} + \lfloor \frac{n}{4} \rfloor) + n + 2, & \text{if } k = 4 \text{ and } n \equiv 1 \pmod 4, \\ (n-k+5)(\frac{n-k+4}{2} + \lfloor \frac{n}{4} \rfloor), & \text{if } k > 4 \text{ and } n \equiv 1 \pmod 4. \end{cases}$$

9 On the $\delta^{(k)}$-Colouring of Graph Products

A graph product of two graphs G and H is a new graph whose vertex set is the Cartesian product of $V(G) \times V(H)$ and for any two vertices (g, h) and (g', h'), the adjacency is defined based on the adjacency (equality, non adjacency) of g and g' and that of h and h'. Depending on how its adjacency is defined, there are 256 different types of graph product. Among the 256 graph products, the concerned investigation considers two most prominent graph products viz. Cartesian product and direct product. The following discussion is on the $\delta^{(k)}$-colouring of Cartesian product and direct product of two graphs. This chapter also consist of the results

obtained for corona product of two graphs. The different graph classes considered here are path graph P_n, cycle graph C_n and complete graph K_n and their different combinations. The optimal $\delta^{(k)}$-colouring that results in minimum number of bad edges for different values of k where $1 \leq k \leq \chi(G) - 1$ is determined.

Definition 9.1. [56, 57] The *Cartesian product* of two graphs G and H is a graph, denoted as $G\Box H$, whose vertex set is $V(G) \times V(H) = \{v_{ij} = (u_i, w_j) : u_i \in V(G), w_j \in V(H)$. Two vertices v_{ij} and v_{kl} are adjacent in $G\Box H$ if either

(i) $u_i = u_k$ and w_j is adjacent to w_l in H; or

(ii) $w_j = w_l$ and v_i is adjacent to v_k in G.

In this study, the $\delta^{(k)}$-colouring of the Cartesian products of different combinations of paths, cycles and complete graphs are investigated. The Cartesian product $P_m\Box P_n$, known as the grid graph, is a bipartite graph, as the Cartesian product of two bipartite graphs is bipartite (see [57]). Hence, its chromatic number is 2 and the possible number of colours for a $\delta^{(k)}$-colouring is $k = 1$. However, colouring the graph with a single colour will lead all the edges to be bad. Hence, the other possible combinations are examined.

A *prism graph*, denoted by $Y_{m,n}$, is the graph Cartesian product $C_n\Box P_m$. The graph $Y_{m,n}$ has mn vertices and $m(2n-1)$ edges. Now, the following discussion is on the $\delta^{(k)}$-colouring of prism graph.

In the Cartesian product of C_n and P_m, the number of bad edges in $C_n\Box P_m$ is m times the number of bad edges in C_n (since there are m copies of C_n say $C_{n,1}, C_{n,2}, \ldots C_{n,m}$ each of order n, in $C_n\Box P_m$) in addition to the number of bad edges between the m C_n's (if there is any). Since $\chi(C_n) = 2$ and $\chi(C_n) = 3$ when n is even and odd respectively, the case of C_n when n is even (as colouring a graph with $k = 1$ colour will result in all its edges to be bad) is excluded and the $\delta^{(k)}$-colouring of $C_n\Box P_m$ when n is odd in the following theorem is determined.

Theorem 9.2. *For $n \equiv 1 \pmod 2$ and $m \geq 2$, the minimum number of bad edges in $Y_{m,n} = C_n \square P_m$ resulting from $\delta^{(k)}$-colouring is given by, $b_2(Y_{m,n}) = 2m - 1$.*

The Cartesian product of two cycles are called *torus grids*. The following theorem discusses the $\delta^{(k)}$-colouring and the minimum number of bad edges in torus grids resulting from the same.

Theorem 9.3. *For $C_n \square C_m$, where $n \geq m$, the minimum number of bad edges resulting from $\delta^{(k)}$-colouring is given by,*

$$b_k(C_n \square C_m) = \begin{cases} 2n, & \text{if } n \text{ is even and } m \text{ is odd,} \\ 2m, & \text{Otherwise.} \end{cases}$$

The $\delta^{(k)}$-colouring and the resultant number of bad edges of the Cartesian product $K_n \square P_m$, $K_n \square C_m$ and $K_n \square K_m$ are determined as explained in the following theorems:

Theorem 9.4. *For the integers n and m where $n \geq m$, the number of bad edges resulting from $\delta^{(k)}$-colouring is given by,*

$$b_k(K_n \square P_m) = \frac{mt(t-1)}{2} + \eta,$$

where η is the total number of bad edges between the m copies of K_n given by

$$\eta = \begin{cases} (m-1)(t-k+1), & \text{if } k \leq \frac{n+1}{2}, \\ 0, & \text{if } k > \frac{n+1}{2}, \end{cases}$$

and $t = n - k + 1$.

Theorem 9.5. *For any two integers n and m, the number of bad edges is given by,*

$$b_k(K_n \square C_m) = \begin{cases} \frac{mt(t-1)}{2} + m(t-k+1), & \text{if } k \leq \lceil \frac{n}{2} \rceil, \\ \frac{mt(t-1)}{2}, & \text{if } k > \lceil \frac{n}{2} \rceil. \end{cases}$$

Theorem 9.6. *For $K_n \square K_m$, where $n \geq m$, the number of bad edges is given by,*

$$b_k(K_n \square K_m) = \frac{mt(t-1)}{2} + \eta(G),$$

where $\eta(G)$ is the total number of bad edges between the m copies of K_n and $\eta(G) = \frac{1}{2}(tm \pmod n)\lfloor \frac{tm}{n} \rfloor \lfloor \frac{tm-n}{n} \rfloor + \frac{1}{2}(n - tm \pmod n)\lfloor \frac{tm-n}{n} \rfloor \lfloor \frac{tm-2n}{n} \rfloor$ and $t = n - k + 1$.

Definition 9.7. *[56, 57] The direct product of G and H is the graph denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$, and for which the vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. Thus,*

1. *$V(G \times H) = \{(g, h) \mid g \in V(G)$ and $h \in V(H)\}$,*
2. *$E(G \times H) = \{(g, h)(g', h') \mid gg' \in E(G)$ and $hh' \in E(G)\}$.*

Other names for the direct product that appears in the literature are *tensor product*, *Kronecker product*, *conjuction*, *cross product* etc. If either of a graph G or H in the direct product is bipartite then their direct product $G \times H$ is bipartite and hence the concerned study does not consist the $\delta^{(k)}$-colouring of path graph and or even cycle and their various combinations. The study is solely focused on cycle graph C_n for odd n and complete graph K_n. Now, $\chi(G \times H)$ is less than or equal to $\min(\chi(G), \chi(H))$ and so when $C_m \times C_n$ and $C_m \times K_n$ are considered, $\chi(C_m \times C_n) = 3$ when both n and m are odd and $\chi(C_m \times K_n) = 3$, when m is odd and hence for these two case the $\delta^{(2)}$-colouring of the same is considered. And, for $K_m \times K_n$, the value of k will be $2 \leq k \leq \min(m, n) - 1$. The direct product is commutative and hence the concerned study focuses on either the $\delta^{(k)}$-colouring of $G \times H$ or $H \times G$.

The following are the results obtained from the $\delta^{(k)}$-colouring of direct product of cycle graph and complete graph with the possible combination.

Theorem 9.8. *For $C_m \times C_n$ where m and n are odd, the minimum number of bad edges obtained from $\delta^{(k)}$-colouring is given by, $b_k(C_m \times C_n) = 2(\min(m, n))$.*

Theorem 9.9. *For $C_m \times K_n$ where m and n are odd, the minimum number of bad edges obtained from $\delta^{(k)}$-colouring is given by, $b_k(C_m \times K_n) = n(n-1)$.*

Theorem 9.10. *For $K_m \times K_n$ where m and n are odd, the minimum number of bad edges obtained from $\delta^{(k)}$-colouring is given by, $b_k(K_m \times K_n) = \frac{n(n-1)(m-k)(m-k+1)}{2}$.*

Definition 9.11. *The corona product of G and H is the graph $G \circ H$ obtained by taking one copy of G, called the centre graph, $|V(G)|$ copies of H, called the outer graph, and making the ith vertex of G adjacent to every vertex of the ith copy of H, where $1 \leq i \leq |V(G)|$ ([58]).*

The corona product is not commutative and hence all the possible combination of $\delta^{(k)}$-colouring of corona product of path graph, cycle graph and complete graphs are taken into consideration.

Theorem 9.12. *For $P_m \circ P_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by, $b_2(P_m \circ P_n) = (m-1)(\frac{1}{2}\lceil\frac{n}{2}\rceil + \lfloor\frac{n}{2}\rfloor)$, for any m and n.*

Theorem 9.13. *For $C_m \circ C_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by,*

$$b_k(C_m \circ C_n) = \begin{cases} \frac{n(3m-1)+4}{4}, & \text{if } k = 2 \text{ and } m \text{ is odd and } n \text{ is even,} \\ \\ \frac{3n}{2}(\frac{n+1}{2}), & \text{if } k = 2 \text{ and } m \text{ is even and } n \text{ is odd.} \end{cases}$$

Theorem 9.14. *For $P_m \circ C_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by,*

$$b_2(P_m \circ C_n) = \begin{cases} \frac{3mn}{4}, & \text{if } m \text{ and } n \text{ are even,} \\ \frac{3mn-n}{4}, & \text{if } m \text{ is odd and } n \text{ is even,} \\ \frac{3m(n+1)}{4}, & \text{if } m \text{ is even and } n \text{ is odd,} \\ \frac{3(m+nm+1)-n}{4}, & \text{if } m \text{ and } n \text{ are odd.} \end{cases}$$

Theorem 9.15. *For $P_m \circ K_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring where $1 \leq k \leq \chi(P_m \circ K_n) - 1$ is given by, $b_k(P_m \circ K_n) = (n-k+1)(\lceil \frac{m}{2} \rceil \frac{(n-k)}{2} + \lfloor \frac{m}{2} \rfloor \frac{(n-k+2)}{2} + 1)$, for all m and n.*

Theorem 9.16. *For $C_m \circ P_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by,*

$$b_2(C_m \circ P_n) = \begin{cases} \frac{m}{2}(n - 1 + \lfloor \frac{n}{2} \rfloor), & \text{if } m \text{ is even and for any } n, \\ \lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor + (n-1)\lfloor \frac{m}{2} \rfloor + 1, & \text{if } m \text{ is odd and for any } n. \end{cases}$$

Theorem 9.17. *For $C_m \circ K_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring where $1 \leq k \leq \chi(C_m \circ K_n) - 1$ is given by,*

$$b_k(C_m \circ K_n) = \begin{cases} \frac{m(n-k+1)4(n-k)+3}{4}, & \text{if } m \text{ is even and } \forall\ n, \\ \frac{m(n-k+1)(n-k+2)+2}{2}, & \text{if } m \text{ is odd and } \forall\ n \end{cases}$$

Theorem 9.18. *For $K_m \circ P_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by,*

$$b_k(K_m \circ P_n) = \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 3 \text{ and for any } m \text{ and } n, \\ \frac{(m-1)(m-2)}{2} + (n-1) + (m-1)\lfloor \frac{n}{2} \rfloor, & \text{if } k = 2 \text{ and for any } m \text{ and } n. \end{cases}$$

Theorem 9.19. *For $K_m \circ C_n$ for any m and n is even, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by,*

$$b_k(K_m \circ C_n) = \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 3, \\ \frac{m(m+n-3)+n+2}{2}, & \text{if } k = 2. \end{cases}$$

Theorem 9.20. *For $K_m \circ C_n$ for any m and n is odd, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by,*

$$b_k(K_m \circ C_n) = \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 4, \\ \frac{(m-2)(m-1)+4}{2}, & \text{if } k = 3, \\ \frac{m(m+n)+n-1}{2}, & \text{if } k = 2. \end{cases}$$

Theorem 9.21. *For $K_m \circ K_n$ for any m and n, the minimum number of bad edges obtained from the $\delta^{(k)}$-colouring is given by,*

$$b_k(K_m \circ K_n) = \begin{cases} (m-k+1)(\frac{m-k}{2} + \frac{(n-k+2)(n-k+1)}{2}) + \frac{(k-1)(n-k+2)(n-k+1)}{2}, & \text{if } k \geq 3, \\ \frac{(n(n-3)+m)(m-1)+2n}{2}, & \text{if } k = 2. \end{cases}$$

10 Conclusion

In the study, on the defective colouring of graphs, the focus was exclusively on the study of $\delta^{(k)}$-colouring of graphs. The $\delta^{(k)}$-colouring of various graph classes,

graph powers and graph products and different graph operations by exhausting all possible $\delta^{(k)}$-colourings for different values of k and determining an optimal $\delta^{(k)}$-colouring that results in minimum number of bad edges for different values of k where $1 \leq k \leq \chi(G) - 1$ are determined. This study also includes few basic yet significant results on the $\delta^{(k)}$-colouring of any graph G. The study can be extended to many graph classes, graph powers and graph products. In the discussion, the number of colour classes, which can have relaxation in proper colouring requirements, is restricted to one. This restriction may also be relaxed ensuring the minimum possible number of bad edges in the given graph. A comparative study on the $\delta^{(k)}$-colouring of graphs by restricting two colour classes to have adjacency between the elements in it with the current definition of $\delta^{(k)}$-colouring can also be done. Studying the relation between $\delta^{(k)}$-colouring and k-defect polynomials for graphs in general is also suggested for further research. Graphs with large $\delta(G)$ motivates the introduction and the study of the notion of the $\delta^{(k)}$-polynomial of a graph. The concept of $\delta^{(k)}$-colouring focuses on the chromatic number of graph, however, finding the $\delta^{(k)}$-colouring of various graph classes for different proper colourings can also be a ground for further research.

Publications

1. M T Ellumkalayil and S Naduvath, "A note on $\delta^{(k)}$-colouring of the Cartesian product of some graph", *Commun. Comb. and Optim.*, DOI:10.22049/CCO.2021.27114.1211, 2021.

2. M T Ellumkalayil and S Naduvath, "On the $\delta^{(k)}$- colouring of generalised Petersen graph", *Discrete Math. Algorithm Appl.*, DOI:10.1142/S1793830921500968, 2021.

3. M T Ellumkalayil and S Naduvath, "On the $\delta^{(k)}$- colouring of powers of helm and closed helm graphs", *Discrete Math. Algorithm Appl.*, DOI: 10.1142/S1793830921501160, 2021.

4. M T Ellumkalayil and S Naduvath, "On the $\delta^{(k)}$- colouring of powers of paths and cycles", *Theory and Applications of Graphs*, 2021. (Accepted)

5. M T Ellumkalayil, L C Samuel, and S Naduvath "Some new results on the $\delta^{(k)}$- colouring of graphs", 2021. (Communicated)

6. M T Ellumkalayil, S Jose, and S Naduvath, "On $\delta^{(k)}$-colouring of some wheel related graphs", 2021. (Communicated)

7. M T Ellumkalayil and S Naduvath, "On $\delta^{(k)}$-colouring of some cycle related graphs", 2021. (Communicated)

Presentations

1. Presented a paper entitled "*On the $\delta^{(k)}$-colouring of powers of paths and cycles*" in an International Conference on Emerging Trends in Graph Theory, CHRIST (Deemed to be University), 2019.

2. Presented a paper entitled "*Some results on the $\delta^{(k)}$-colouring of powers of paths, cycles and generalised Petersen graphs*" in an International Conference of Discrete Mathematics (ICDM 2019) organised by the Academy of Discrete Mathematics and Applications (ADMA) jointly with CHRIST (Deemed to be University), 2019.

3. Presented a paper entitled "*Study on $\delta^{(k)}$-colouring of graphs*" in a National conference on Emerging trends in Graph Theory and its Applications, CHRIST (Deemed to be University), 2020.

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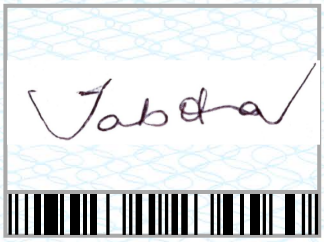

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